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A qualitative investigation of the oscillations of a pendulum with a periodically varying length and a mathematical model of a swing☆

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Abstract

The behaviour of the amplitude-frequency characteristics of families of periodic solutions, produced from the equilibrium position of a system, is established by a qualitative investigation of the equation of the oscillations of a pendulum, the length of which is an arbitrary periodic function of time. The non-local conditions for their stability and instability, expressed in terms of the amplitude and frequency of the oscillations, are obtained. The results are used when discussing the parametric and self-excited oscillatory model of a swing. In the parametric model the length of a swing is a specified periodic function of time, and in the self-excited oscillatory model it is a function of the phase coordinates of the system. For an appropriate choice of these functions, both systems have a common periodic solution. It is shown that the parametric model leads to an erroneous conclusion regarding the instability of the periodic mode, which is in fact realized in the oscillations of a swing, whereas the self-excited oscillatory model indicates its stability.

The majority of investigations of the oscillations of a pendulum, the length of which varies periodically (for example, Refs 1–7) have been carried out by constructive, mainly asymptotic, methods, on the assumption that the variation of the length of the pendulum or the amplitude of its oscillations are small (i.e. the system is quasi-conservative or quasi-linear). The question of the permissible value of these small parameters and, consequently, of the range of applicability of the results obtained, as a rule, remains open.

In this paper we carry out a rigorous qualitative investigation of this system, free from these assumptions; the majority of the results obtained hold for any law of variation of the length of the pendulum.

1. Formulation of the problem

Preliminary results. The equation of the oscillations of a pendulum, the length of which varies periodically, can be written in the form

$$[l^2(\omega t)\dot{x}]^{\boldsymbol{\cdot}} + \mu\varphi(x, \dot{x}) + gl(\omega t)\sin x = 0 \tag{1.1}$$

Here $x = x(t, \mu)$ is the angular coordinate $\dot{x} = dx/dt$, $l(\omega t) = l(\omega t + 2\pi)$ is the distance from the suspension point to the centre of gravity (the length) of the pendulum, g is the acceleration due to gravity, $\mu > 0$ is a parameter, and the

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function $\varphi(x, \dot{x})$ describes non-conservative forces, which are assumed to be dissipative, i.e.

$$\varphi_{\dot{x}}(x, \dot{x}) = \partial\varphi(x, \dot{x})/\partial\dot{x} > 0 \text{ for all } x \text{ in } \dot{x} \tag{1.2}$$

When there are no dissipative forces ($\mu = 0$), Eq. (1.1) takes the form

$$[l^2(\omega t)\dot{x}]^{\bullet} + gl(\omega t)\sin x = 0 \tag{1.3}$$

We will investigate the relation between the periodic solutions of Eqs (1.1) and (1.3). Suppose $x(t) - x(t+T)$ is the solution of Eq. (1.3); the following equation in variations corresponds to it

$$[l^2(\omega t)\dot{y}]^{\bullet} + r(\omega t)y = 0; \quad r(t) = r(t+T) = gl(\omega t)\cos x(t) \tag{1.4}$$

We will assume that the multipliers $\rho_i (i = 1, 2)$ of Eq. (1.4) are complex. Then, as is well known, $|\rho_i| = 1$, in which case Eq. (1.4) is strongly stable (i.e. stability is maintained for fairly small perturbations $l(\omega t)$ and $r(\omega t)$). Since, by virtue of the inequalities $\rho_i \neq 1$ Eq. (1.4) has no T-periodic solutions, them, by Poincaré's theorem on the continuation of solutions with respect to a parameter, Eq. (1.1) has a unique solution $x(t, \mu) = x(t+T, \mu)$ for sufficiently small μ, such that $x(t, 0) = x(t)$.

We will establish a relation between the stability of the solutions $x(t)$ and $x(t, \mu)$. The stability of the solution $x = x(t, \mu)$ is defined by the equation in variations

$$\begin{gathered}[l^2(\omega t)\dot{y}]^{\bullet} + \mu b(t, \mu)\dot{y} + r(t, \mu)y = 0 \\ b(t, \mu) = \varphi_{\dot{x}}(x, \dot{x}), \quad r(t, \mu) = gl(\omega t)\cos x + \mu\varphi_x(x, \dot{x})\end{gathered} \tag{1.5}$$

Since the multipliers $\rho_i(\mu)$ are continuous in μ, for sufficiently small μ they remain complex-conjugate, and consequently $|\rho_1(\mu)| = |\rho_2(\mu)|$. As is well known, $\rho_i(\mu)$ are the roots of the characteristic equation

$$\rho^2 - 2A(\mu)\rho + B(\mu) = 0; \quad A(\mu) = [x_1(T, \mu) + x_2(T, \mu)]/2$$

where $x_1(t, \mu)$ and $x_2(t, \mu)$ are the solutions of Eq. (1.5), which satisfy the conditions

$$x_1(0, \mu) = 1, \quad \dot{x}_1(0, \mu) = 0; \quad x_2(0, \mu) = 0, \quad \dot{x}_2(0, \mu) = 1$$

while $B(\mu)$ is defined by Liouville's formula

$$B(\mu) = \exp\left(-\frac{\mu}{2}\int_0^T b(t, \mu)dt\right)$$

With condition (1.2) $b(t, \mu) > 0$, and hence

$$\rho_1(\mu)\rho_2(\mu) = B(\mu) < 1 \tag{1.6}$$

and, consequently,

$$|\rho_1(\mu)| = |\rho_2(\mu)| < 1 \tag{1.7}$$

When μ increases this inequality and, consequently, the uniqueness of the continuation of $x(t, \mu)$ with respect to the parameter μ, is retained, so long as the multipliers $\rho_i(\mu)$ remain complex. The multiplier $\mu_1(\mu)$ may reach the point $\mu = 1$ or $\mu = -1$, only moving along the real axis; in this case, by virtue of the inequalities (1.6) and $\rho_i(\mu) \neq 0$ the multiplier $\rho_2(\mu)$ lies correspondingly in the interval (0, 1) or (−1,0). In the first case further continuation of the family $x(t, \mu)$ with respect to μ is, generally speaking, impossible; in the second case at this point a family of 2T-periodic solutions usually branches out from the T-periodic family considered.

Hence, the high stability of the solution $x(t)$ of Eq. (1.3) guarantees the existence and asymptotic stability of the family of solutions $x(t, \mu)$ of Eq. (1.1) up to the boundary of its existence or up to the bifurcation point.

A similar situation occurs if the solution $x(t)$ is unstable and $\rho_i \neq \pm 1$. Here the multipliers are real, and one of them is greater than unity in modulus. Consequently, when μ increases the multipliers move along the real axis; instability of the family $x(t, \mu)$ is maintained so long as $\rho_i(\mu) \neq \pm 1$.

Fig. 1.

Thus, the existence and stability of the family of periodic solutions $x(t, \mu)$ of Eq. (1.1) are determined by the existence and stability of the corresponding solution for $\mu = 0$, so henceforth we will confine ourselves to investigating Eq. (1.3) when only the parameter ω varies.

We will consider families of periodic solution $x_k(t, \mu)$ of Eq. (1.3), produced from the equilibrium position of the system ($x_k(t, \mu) \to 0$ as $\omega \to \omega_k$). The quantities ω_{2k-1} and ω_{2k} are the boundaries of the k-th region of parametric resonance of the linearized system, i.e. when $\omega \in (\omega_{2k-1}, \omega_{2k})$ the linearized equation is unstable, and when $\omega \in (\omega_{2k}, \omega_{2k+1})$ it is stable. The inequality $\omega_{2k-1} \geq \omega_{2k} > \omega_{2k+1}$ holds, and in the generic case $\omega_{2k-1} > \omega_{2k}$ (Fig. 1).

Assuming $\tau = \omega t$, $\lambda = 1/\omega$, we reduce Eq. (1.3) to the form

$$[l^2(\tau)x']' + \lambda^2 g l(\tau)\sin x = 0; \quad l(\tau) = l(\tau + 2\pi) \tag{1.8}$$

where the prime denotes differentiation with respect to τ.

For infinitely small $|x|$ Eq. (1.8) becomes the corresponding linearized equation

$$[l^2(\tau)x']' + \lambda^2 r_0(\tau)x = 0; \quad r_0(\tau) = gl(\tau) \tag{1.9}$$

Suppose $x(\tau)$ is a periodic solution of Eq. (1.8). It can be shown by direct substitution that the function $x(\tau)$ also satisfies the linear equation

$$[l^2(\tau)x']' + \lambda^2 r_1(\tau)x = 0; \quad r_1(\tau) = gl(\tau)\sin x(\tau)/x(\tau) \tag{1.10}$$

The following equation in variations corresponds to the solution $x(\tau)$

$$[l^2(\tau)y']' + \lambda^2 r_2(\tau)y = 0; \quad r_2(\tau) = gl(\tau)\cos x(\tau) \tag{1.11}$$

We will recall some facts of the theory of the stability of equations with periodic coefficients,[8] which we will use below. Consider the equation

$$[l^2(\tau)x']' + \lambda^2 r(\tau)x = 0; \quad l(\tau) = l(\tau + 2\pi), \quad r(\tau) = r(\tau + 2\pi) \tag{1.12}$$

Assuming

$$x = z_1, \quad l^2 x' = \lambda z_2$$

we reduce Eq. (1.12) to canonical form

$$J\mathbf{z}' = \lambda H(\tau)\mathbf{z}$$
$$\mathbf{z} = \left\| \begin{matrix} z_1 \\ z_2 \end{matrix} \right\|, \quad J = \left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\|, \quad H(\tau) = \left\| \begin{matrix} r(\tau) & 0 \\ 0 & l^{-2}(\tau) \end{matrix} \right\| \tag{1.13}$$

The multipliers of Eq. (1.13) are the roots of the characteristic equation

$$\rho^2 - 2A(\lambda)\rho + 1 = 0; \quad A(\lambda) = [z_{11}(2\pi, \lambda) + z_{22}(2\pi, \lambda)]/2 \tag{1.14}$$

where $A(\lambda)$ is the characteristic function, and ($z_{11}(\tau, \lambda)$ and $z_{22}(\tau, \lambda)$ are the diagonal elements of the matrix $Z(\tau, 0, \lambda)$ of Eq. (1.13), where $Z(0, 0, \lambda) = I$ is the unit matrix). The multipliers are complex and, consequently, Eq. (1.13) is strongly stable provided

$$|A(\lambda)| < 1 \tag{1.15}$$

We will assume that $r(\tau) \geq 0$, in which case the matrix $H(\tau)$ is negative-definite. For the equation

$$J\mathbf{z}' = [H_0(\tau) + \lambda H(\tau)]\mathbf{z}$$

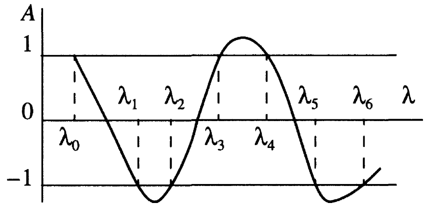

Fig. 2.

the function $A(\lambda)$ has the form shown in Fig. 2 (Ref. 8, Chapter 7, Fig. 37). It is easy to show that $\lambda_0 = 0$ for Eq. (1.13) (where $H_0(\tau) = 0$). In the generic case $\lambda_{2k-1} < \lambda_{2k}$, but in principle the equalities $\lambda_{2k-1} < \lambda_{2k}$ are possible.

It follows from Fig. 2 and condition (1.15) that Eq. (1.13) is strongly stable when $\lambda \in (\lambda_{2k}, \lambda_{2k+1})$ and unstable when $\lambda \in (\lambda_{2k+1}, \lambda_{2k+2})$, $k = 0, 1, 2, \ldots$.

Since $A(\lambda) = 1$ when $\lambda = \lambda_k$, $k = 3, 4, 7, 8 \ldots$, we have $\rho_1 = \rho_2 = 1$, and hence Eq. (1.13) has the solution

$$\dot{\mathbf{z}}(\tau) = \mathbf{z}(\tau + 2\pi)$$

When $\lambda = \lambda_k$, $k = 1, 2, 5, 6, \ldots$ we have $A(\lambda) = -1$, $\rho_i = -1$, and therefore

$$\mathbf{z}(\tau) = -\mathbf{z}(\tau + 2\pi) = \mathbf{z}(\tau + 4\pi)$$

Hence, λ_k are the eigenvalues of the periodic and antiperiodic boundary-value problem for Eq. (1.13).

These boundary-value problems are self-conjugate, and hence positive eigenvalues decrease as $r(\tau)$ increase. Since $1 > \sin x/x > \cos x$ when $|x| < \pi$, $x \neq 0$, then, almost everywhere

$$r_0(\tau) > r_1(\tau) > r_2(\tau) \tag{1.16}$$

whence it follows that

$$\lambda_k(r_0) < \lambda_k(r_1) < \lambda_k(r_2), \quad k = 1, 2, \ldots \tag{1.17}$$

where $\lambda_k(r_0)$, $\lambda_k(r_1)$ and $\lambda_k(r_2)$ correspond to Eqs (1.9), (1.10) and (1.11).

2. The existence and stability of periodic oscillations of a pendulum

As was noted above, in the generic case $\omega_{2k-1} < \omega_{2k}$. Then, in a sufficiently small neighbourhood of the equilibrium position a unique family $x_k(t,\omega)$ of periodic solutions of Eq. (1.3) exists such that $x_k(t, \omega) \to 0$ as $\omega \to \omega_k$. The generating solutions of the linearized system are pairwise periodic and antiperiodic; we will consider families which preserve these properties, i.e.

$$x_k(t, \omega) = x_k(t + 2\pi/\omega, \omega), \quad k = 3, 4, 7, 8, \ldots \tag{2.1}$$

and

$$x_k(t, \omega) = -x_k(t + 2\pi/\omega, \omega), \quad k = 1, 2, 5, 6, \ldots \tag{2.2}$$

The purpose is to obtain non-local criteria of the existence and stabilities of these families. When deriving these criteria we will consider the corresponding families of solutions of Eq. (1.8), i.e.

$$x_k(\tau, \lambda) = x_k(\tau + 2\pi, \lambda) \text{ and } x_k(\tau, \lambda) = -x_k(\tau + 2\pi, \lambda)$$

They serve as solutions of the boundary-value problems for Eq. (1.8) with the conditions

$$x(0, \lambda) = \chi x(2\pi, \lambda), \quad x'(0, \lambda) = \chi x'(2\pi, \lambda) \tag{2.3}$$

for $\chi = +1$ and $\chi = -1$ respectively.

As is well known, these solutions are uniquely continuable with respect to the parameter λ, if the corresponding boundary-value problems for the equation in variations (1.11) (where $x = x_k(\tau, \lambda)$) do not have non-trivial solutions, i.e.

$$\lambda \neq \lambda_k(r_2), \quad k = 3, 4, 7, 8, \ldots \tag{2.4}$$

or

$$\lambda \neq \lambda_k(r_2), \quad k = 1, 2, 5, 6, \ldots \tag{2.5}$$

Suppose $A_k(\omega) = \max_t |x_k(t, \omega)|$ is the amplitude of the solution $x_k(t, \omega)$.

The following theorem gives the sufficient conditions of continuability of the solutions $x_k(t,\omega)$ ($x_k(t, \omega) = 0$) with respect to ω.

Theorem 1. *The solutions $x_k(t,\omega)$ are uniquely continuable with respect to the parameter ω in the direction of its decrease, so long as*

$$\omega > \omega_2 \ (k = 1) \tag{2.6}$$

$$\omega > \omega_4, \quad A_3(\omega) \leq \pi/2 \ (k = 3) \tag{2.7}$$

$$\omega > \omega_{k+1}, \quad \omega^2 \cos A_k(\omega) > \omega_{k-3}^2 \ (k = 5, 7, \ldots) \tag{2.8}$$

$$\omega > \omega_{k+3}, \quad \omega^2 \cos A_k(\omega) > \omega_{k-1}^2 \ (k = 2, 4, \ldots) \tag{2.9}$$

Proof. We will first show that, for sufficiently small amplitudes, the parameter λ increases along the family of solutions $x_k(\tau, \lambda)$. Since the solution $x_k(\tau, \lambda)$ satisfies Eq. (1.10) and boundary conditions (2.3), the value of λ is equal to the eigenvalue of the corresponding boundary-value problem, i.e. $\lambda = \lambda_j(r_1)$ for certain $j = j(\lambda)$. Since $x_k(\tau, \lambda) \to 0$ and $r_1(\tau) \to r_0(\tau)$ when $\lambda \to \lambda_k(r_0)$, then $\lambda_j(r_1) \to \lambda_j(r_0)$, and hence $j(\lambda_1) = k$ for small $|\lambda - \lambda_k|$. But, by virtue of inequalities (1.17)

$$\lambda = \omega^{-1} = \lambda_k(r_1) > \lambda_k(r_0) = \omega_k^{-1}$$

which it was required to prove. □

We will show that inequalities (2.6)–(2.9) guarantee that conditions (2.4) and (2.5) are satisfied and thereby the non-local continuity of the solution $x_k(\tau, \lambda)$ with respect to λ. On account of inequalities (1.17) $\lambda_k(r_1) \to \lambda_k(r_2)$, and hence conditions (2.4) or (2.5) are necessarily satisfied, so long as

$$\lambda = \lambda_k(r_1) \text{ and } \lambda > \lambda_k^-(r_2) \tag{2.10}$$

where $\lambda_k^-(r_2)$ is the closest eigenvalue of the corresponding problem on the left of $\lambda_k(r_2)$.

We will first consider the solutions $x_k(\tau, \lambda)$ when $k = 5, 7, \ldots$. As λ increases the first condition of (2.10) may break down if $\lambda_k(r_1) = \lambda_{k+1}(r_1)$ for certain $\lambda = \lambda_*$, and as a result when $\lambda \to \lambda_*$ instead of $\lambda = \lambda_k(r_1)$ the equality $\lambda = \lambda_{k+1}(r_1)$ may be satisfied, which does not guarantee the required inequality $\lambda < \lambda(r_2)$. By virtue of the first condition of (2.8) and inequalities (1.17) we have

$$\lambda < \lambda_{k+1} = \lambda_{k+1}(r_0) < \lambda_{k+1}(r_1)$$

i.e. the first condition of (2.9) is satisfied. As can be seen from Fig. 2, $\lambda_k^-(r_2) = \lambda_{k-3}(r_2)$ when $k = 5, 7, \ldots$; taking into account the fact that $\lambda = 1/\omega$, $\lambda_{k-3}(r_0) = 1/\omega_{k-3}$, we have from the second condition of (2.8)

$$r_2(\tau) = gl(\tau)\cos x_k(\tau) > gl(\tau)\cos A_k(\omega) > gl(\tau)\lambda_{k-3}^2(r_0)/\lambda^2 \tag{2.11}$$

It is obvious that $\lambda_k(c^2 r_2) = \lambda_k(r_2)/c$ for any $c > 0$, and hence, taking inequality (2.11) into account, we obtain the required inequality

$$\lambda_{k-3}(r_2) < \lambda_{k-3}(gl(\tau)\lambda_{k-3}^2(r_0)/\lambda^2) = \lambda$$

When $k = 1, 3$ there are no positive eigenvalues on the left of $\lambda_k(r_2)$, and hence conditions (2.4) and (2.5) are ensured by the first inequality of (2.6). Nevertheless, the condition $A_3(\omega) \leq \pi/2$ is included in the system of conditions (2.7), because it guarantees the inequality $r_2(\tau) \geq 0$ (otherwise it is possible that $\lambda_0 > 0$ (Fig. 2), so that when the first condition of (2.7) is satisfied the solution $x_3(\tau, \lambda)$ is only continuable up to λ_0).

The theorem for the solutions $x_k(\tau, \lambda)$ for $k = 2, 4, 6, \ldots$ is proved similarly. Here the eigenvalues closest to the left and the right of $\lambda_k(r_2)$ are equal to $\lambda_{k-1}(r_2)$ and $\lambda_{k+3}(r_2)$ (Fig. 2), and therefore conditions (2.8) are replaced by conditions (2.9). The theorem is proved.

Note the following fact. Solutions (2.1) have a minimum period $T = 2\pi/\omega$; under the conditions of Theorem 1 the corresponding multipliers $\rho_i \neq 1$ and consequently, the continuation of these solutions with respect to ω is unique. However, they do not exclude the equalities $\rho_1 = \rho_2 = -1$; in this case solutions with period 2T may branch out from the family considered.

The solutions (2.2) have a period 2T; however, taking into account the equality $\cos x = \cos(-x)$ it is easy to verify that the minimum period of the coefficients of the corresponding equation in variations is equal to T. The conditions of the theorem exclude the equality $\rho_1 = \rho_2 = -1$, but do not guarantee the conditions $\rho_i \neq 1$, and hence for certain ω solutions with period 2T, not satisfying relation (2.2), may branch out from the family considered.

Since the trajectories of Eq. (1.3) cannot pass through the singular points $x = \pm\pi$, $x' = 0$, the amplitude-frequency characteristics of the solutions $x_k(t, \omega)$ satisfy the inequality $A_k(\omega) < \pi$. If the family $x_k(t, \omega)$ is continuable to $\omega = 0$, then $A_k^* = \lim A_k(\omega) = \pi$ as $\omega \to 0$. In fact, when $A_k^* < \pi$ in Eq. (1.10) $r_1(\tau) \geq gl(\tau)(\sin A_k^*)/A_k^* > 0$, and hence the corresponding eigenvalue $\lambda_1(r_1) < \delta$ and, consequently, $\omega = \lambda_1^{-1}(r_1) > \delta^{-1}$.

We will determine the nature of the solutions considered.

Theorem 2. *The solutions* $x_{2k-1}(t, \omega)$ and $x_{2k}(t, \omega)$ *for* $k = 1, 2, \ldots$ *have k zeros and k extrema in* $[0, 2\pi/\omega]$.

Proof. As was shown above, the generating solutions $x_{2k-1}(t)$ and $x_{2k}(t)$ for $k = 3, 4, 7, 8, \ldots$ are the eigenfunctions of the periodic boundary-value problem for Eq. (1.9). For the latter, the assertion of the theorem and the number of zeros follows from the Sturm-Liouville theory. For continuous continuation of the family $x_k(t, \dot{\omega})$ with respect to the parameter ω the number of zeros is maintained, otherwise $x_k(t_*, \omega_*) = x_k(t_*\omega_*) = 0$ for certain t_* and ω_*, which is possible provided $x_k(t, \omega_*) \equiv 0$.

Suppose t_1 and t_2 are neighbouring zeros of the solution $x_k(t, \omega)$, and when $t_* \in (t_1, t_2)$ it has an extremum. Taking into account the fact that $x(t_*) = 0$, we obtain

$$l^2(t)\dot{x}(t) = -\int_{t_*}^{t} gl(s)\sin x(s)ds$$

Since $\sin x(t) > 0$ in $(t_1, t_2$ and $l(t) > 0$, then $x(t) > 0$ in (t_1, t_*) and $\dot{x}(t) < 0$ in (t_*, t_2). Hence, there is only one extremum between neighbouring zeros of the solution $x(t)$, i.e. $x_k(t, \omega)$ varies monotonically between neighbouring extremum values. Taking the previous result into account we obtain that the solutions $x_{2k-1}(t, \omega)$ and $x_{2k}(t, \omega)$ have k extrema in $[0, 2\pi/\omega]$. The theorem is proved.

We will estimate the stability of the solutions. □

Theorem 3. *For conditions* (2.9) *the solutions* $x_k(t, \omega)$ *are unstable for* $k = 2, 4, \ldots$, *while for* $k = 1, 3, \ldots$ *they are stable, so long as*

$$\omega > \omega_2, \quad A_1(\omega) \leq \pi/2 \ (k = 1) \tag{2.12}$$

and

$$\omega > \omega_{k+1}, \quad \omega^2 \cos A_k(\omega) > \omega_{k-1}^2 \ (k = 3, 5, \ldots) \tag{2.13}$$

Proof. As was noted above, Eq. (1.13) is stable when $\lambda \in (\lambda_{2k}, \lambda_{2k+1})$ and unstable when $\lambda \in (\lambda_{2k+1}, \lambda_{2k+2})$, $k = 0, 1, 2, \ldots$. When proving Theorem 1 we showed that, for condition (2.9) for the solutions $x_k(t, \omega)$ for $k = 2, 4, \ldots$ we have the condition $\lambda \in (\lambda_{2k+1}, \lambda_{2k+2})$, which also proves their instability.

For the solutions $x_k(t, \omega)$ for $k = 5, 7, \ldots$ the second inequality of (2.8) guarantees the condition $\lambda \in (\lambda_{2k}, \lambda_{2k+3})$. As can be seen from the proof, to obtain the required stability condition $\lambda \in (\lambda_{2k}, \lambda_{2k+1})$ it is sufficient to replace ω_{k-3} by ω_{k-1} in this inequality. Condition (2.13) thereby obtained also guarantees the stability of the solution $x_3(t, \omega)$. For the solution $x_1(t, \omega)$ the first condition of (2.12) ensures the inequality $\lambda < \lambda_1$, while the second condition ensures the non-negativity of $r_2(\tau)$. The theorem is proved. □

The results obtained above hold for any law of variation of the length of the pendulum. We will now assume that $l(\omega t)$ is an even function, i.e.

$$l(\tau) = l(-\tau) = l(\tau + 2\pi) \tag{2.14}$$

For example, if the length varies harmonically, we can assume $l(\tau) = l_0 + \varepsilon\cos\tau$.

We will assume that the conditions of Theorem 1 are satisfied in addition to the condition (2.14).

Theorem 4. *One of the families $x_{2k-1}(t, \omega)$ and $x_{2k}(t, \omega)$, $k = 1, 2, \ldots$ is even and the other is odd.*

Proof. We first note that if $x(\tau)$ is a unique periodic or antiperiodic solution of Eq. (1.8), it is even or odd. In fact, we put $x(\tau) = x_1(\tau) + x_2(\tau)$, where $x_1(\tau) = x_1(-\tau)$ and $x_2(\tau) = -x_2(-\tau)$. Substituting this expression into Eq. (1.8) and taking condition (2.14) into account, we obtain that each of the functions $x_1(\tau)$ and $x_2(\tau)$ separately satisfies this equation, and hence $x(\tau) = x_1(\tau)$ or $x(\tau) = x_2(\tau)$.

We will show that the generating solutions $x_{2k-1}(\tau)$ and $x_{2k}(\tau)$, $k = 1, 2, \ldots$ cannot be simultaneously even or odd. In fact, the even solutions are eigenfunctions of the boundary-value problem for Eq. (1.10) when $x'(0) = x'(2\pi) = 0$. By the Sturm-Liouville theory, any two such functions have a different number of zeros; nevertheless, in view of Theorem 2, for the solutions $x_{2k-1}(\tau)$ and $x_{2k}(\tau)$ the number of zeros is the same. It can similarly be shown that both solutions $x_{2k-1}(\tau)$ and $x_{2k}(\tau)$ cannot be odd. The theorem is proved. □

3. Fundamental parametric oscillations

We will consider the solutions of the families (2.2) for $k = 1, 2$, corresponding to the fundamental parametric resonance, in more detail (note that the first qualitative results of this kind were obtained in Ref. 9).

It follows from Theorem 3 that these solutions have two zeros and two extrema in the half-interval $[0, 4\pi/\omega)$. The following theorem shows that these solutions are continuable with respect to ω down to $\omega = 0$.

Theorem 5. *For condition (2.14) the solutions of the family (2.2), $k = 1, 2$ are uniquely continuable with respect to ω down to $\omega = 0$. The amplitudes $A_k(\omega)$ decrease monotonically in $(0, \omega_k)$.*

Proof. According to Theorem 4 the functions of one of the families $x_1(\tau, \lambda)$ and $x_2(\tau, \lambda)$ are even, and the other is odd. Since the even antiperiodic solution becomes odd when the origin of coordinates is shifted by π, it is sufficient to prove the theorem for odd solutions; we will denote them by $x(\tau, \lambda)$.

Since the function $x(\tau, \lambda)$ has two zeros in $[0, 4\pi)$, it retains its sign in $(0, 2\pi)$ and satisfies the conditions

$$x(0) = x(2\pi) = 0 \tag{3.1}$$

Consequently, $\lambda = \lambda_1(r_1)$ and $x(\tau, \lambda)$ are the first eigenvlaue and the eigenfunction of the boundary-value problem for Eq. (1.10) with conditions (3.1). By virtue of inequalities (1.17) $\lambda_1(r_1) < \lambda_1(r_2)$, and consequently the condition of continability of $x(\tau, \lambda)$ with respect to the parameter λ ($\lambda \neq \lambda_k(r_2)$, $k = 1, 2, \ldots$) is satisfied when $\lambda \in (0, \infty)$.

Without loss of generality, we will assume that $x(\tau, \lambda) > 0$ when $\tau \in (0, 2\pi)$. According to Theorem 3 the function $x(\tau, \lambda)$ has one extremum in $(0, 2\pi)$; since $l(\tau) = l(2\pi - \tau)$, it is easy to show that $x(\tau, \lambda) = x(2\pi - \tau, \lambda)$, and hence this extremum is reached when $\tau = \pi$, i.e. the amplitude of the solution $A(\lambda) = x(\pi, \lambda)$.

Differentiating Eq. (1.8) with respect to the parameter λ, we obtain that the function $x_\lambda(\tau, \lambda) = \partial x(\tau, \lambda)/\partial\lambda$ satisfies the equation

$$[l^2(\tau)(x_\lambda)']' + \lambda^2 r_2(\tau) x_\lambda = -2\lambda g l(\tau)\sin x(\tau, \lambda) \tag{3.2}$$

Taking into account the fact that $x_\lambda(\tau, \lambda)$ satisfies condition (3.1), the solution of Eq. (3.2) in $(0, 2\pi)$ can be represented in the form

$$x_\lambda(\tau, \lambda) = -2\lambda g \int_0^{2\pi} G(\tau, s) l(s) \sin x(s, \lambda) ds \tag{3.3}$$

where $G(\tau, s)$ is Green's function of the corresponding boundary-value problem (1.11), (3.1).

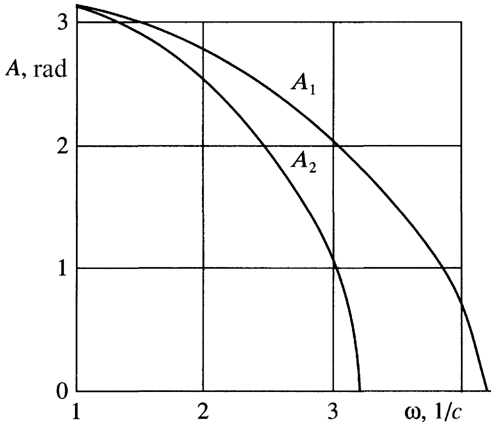

Fig. 3.

As is well known, the function $G(\tau, s)$ is continuous when $\tau, s \in [0, 2\pi]$, and satisfies Eq. (1.11)when $\tau \neq s$ and the conditions

$$G(0, s) = G(2\pi, s) = 0, \quad G'(s+0, s) - G'(s-0, s) = 1/l^2(s) \tag{3.4}$$

It can be shown by a direct check that

$$G(\tau, s) = -\frac{x(2\pi, s)}{x(2\pi, 0)}x(\tau, 0) + \tilde{G}(\tau, s); \quad \tilde{G}(\tau, s) = \begin{cases} 0 \text{ for } \tau < s \\ x(\tau, s) \text{ for } \tau > s \end{cases} \tag{3.5}$$

where $x(\tau, s)$ is the solution of Eq. (1.11), which satisfies the conditions $x(s, s) = 0$, $x'(s, s) = 1/l^2(s)$.

By virtue of the condition λ, $\lambda_1(r_2)$ any solution of Eq. (1.11) in $[0, 2\pi]$ has no more than one zero, and hence $x(\tau, s) > 0$ when $0 \leq s < \tau < 2\pi$. Taking this inequality into account, we obtain from relations (3.5) and (3.4) that $G(\tau, s) < 0$ when $0 \leq s < \tau < 2\pi$ and, consequently, $x_\lambda(\tau, \lambda) > 0$ ($x(\tau, \lambda) > 0$). Hence, the solution $x(\tau, \lambda)$ increases with respect to λ, i.e. the corresponding solution $x(t, \omega)$ decreases with respect to ω in $(0, 2\pi/\omega)$, and hence its amplitude $A(\omega) = x(\pi/\omega, \omega)$ also decreases. The theorem is proved. □

As shown above, $A_k^* = \lim A_k(\omega) = \pi$ when $\omega \to 0$, $k = 1, 2$. Hence, the amplitude-frequency characteristics of the solutions considered have the form shown in Fig. 3 (here we have taken $l(\omega t) = (3 - 0.5 \cos \omega t)$ m and $g = 9.81 \text{ m/s}^2$).

We emphasise that the theorem does not exclude branching of the families considered for certain ω; the uniqueness of the continuation merely denotes that the new branches do not possess this symmetry (i.e. they are neither even nor odd).

We will assume in addition that the length of the pendulum varies monotonically between the extremum values; without loss of generality we will assume that

$$l'(\tau) \geq 0 \text{ when } \tau \in (0, \pi) \tag{3.6}$$

Note that this condition uniquely defines the origin of τ.

We will investigate the stability of the above solutions.

Theorem 6. *When conditions* (2.14) and (3.6) *are satisfied the solutions $x_1(t, \omega)$ are odd and $x_2(t, \omega)$ are even. The solutions $x_1(t, \omega)$ are stable when $A_1(\omega) \leq \pi/2$, and $x_2(t, \omega)$ are unstable for all ω.*

Proof. Suppose $x(\tau, \lambda)$ is odd from the solutions $x_1(\tau, \lambda)$ and $x_2(\tau, \lambda)$. As shown above, $x(\tau, \lambda) = x(2\pi - \tau, \lambda)$, and hence, in the corresponding equation in variations (1.11)

$$r_2(\tau) = r_2(-\tau), \quad r_2(\tau) = r_2(2\pi - \tau) \tag{3.7}$$

In view of the first equality of (3.7) the even eigenfunction corresponds to the one of the eigenvalues $\lambda_1(r_2)$ and $\lambda_2(r_2)$ of the antiperiodic boundary-value problem for Eq. (1.11), while the odd eigenfunction corresponds to the other.

We will denote them by $u(\tau)$ and $\upsilon(\tau)$ respectively. In view of the second equality of (3.7) we have

$$u(\tau) = -u(2\pi-\tau), \quad \upsilon(\tau) = \upsilon(2\pi-\tau)$$

Hence, the functions $u(\tau)$ and $\upsilon(\tau)$ satisfy the relations

$$[l^2(\tau)u']' + \eta_1^2 r_2(\tau)u = 0, \quad u'(0) = u(\pi) = 0 \tag{3.8}$$

and

$$[l^2(\tau)\upsilon']' + \mu_1^2 r_2(\tau)\upsilon = 0, \quad \upsilon(0) = \upsilon'(\pi) = 0 \tag{3.9}$$

where $\mu_1 = \lambda_1$, $\eta_1 = \lambda_2$ or $\eta_1 = \lambda_1$, $\mu_1 = \lambda_2$.

Without loss of generality we will assume that

$$x(\tau, \lambda) > 0, \quad u(\tau) > 0, \quad \upsilon(\tau) > 0 \text{ in } (0, \pi) \tag{3.10}$$

Then

$$u'(\tau) < 0, \quad \upsilon'(\tau) > 0 \text{ in } (0, \pi) \tag{3.11}$$

We will show that the following condition is satisfied for the solution considered

$$\lambda < \lambda_1(r_2) = \min(\mu_1(r_2), \eta_1(r_2))$$

which, in accordance with Fig. 2, guarantees the inequality $|A(\lambda)| < 1$ and thereby the stability of the solution $x(\tau, \lambda)$. Since the function $x(\tau, \lambda)$ satisfies Eq. (1.10), and by the conditions $x(0, \lambda) = x(2\pi\lambda) = 0$ it also has no zeros in $(0, 2\pi)$, we have $\lambda = \mu_1(r_1)$. By virtue of condition (1.17) $\lambda = \mu_1(r_1) < \mu_1(r_2)$; we will show that $\lambda < \eta_1(r_2)$.

In Eq. (1.8) we put $x = x(\tau, \lambda)$ and differentiate with respect to τ. We obtain that the function $y = x'(\tau, \lambda)$ satisfies the equation

$$[l^2(\tau)y']' + \lambda^2 r_2(\tau)y = -2[l(\tau)l'(\tau)y]' - \lambda^2 g l'(\tau)\sin x(\tau, \lambda) \tag{3.12}$$

We multiply both sides of this equation by $u(\tau)$ and Eqs (3.8) by $y(\tau)$ and integrate the difference of the relations obtained in the limits from 0 to π. After integration by parts we obtain

$$(\lambda^2 - \eta_1^2)\int_0^\pi r_2 u y d\tau = [l^2 u' y - l^2 y' u - 2ll' y u]_0^\pi + \int_0^\pi (2ll' y u' - \lambda^2 g l' u \sin x) d\tau \tag{3.13}$$

Since $x(\tau, \lambda) = -x(-\tau, \lambda)$, we have $y'(0) = x''(0) = 0$. Taking boundary conditions (3.8) and (3.9) into account as well as the equality $l'(0) = l'(\pi) = 0$, we obtain that the term outside the integral in relation (3.13) is equal to zero. For conditions (3.6), (3.10), (3.11) and $|x(\tau, \lambda)| \le \pi/2$, the integrand on the left-hand side of relation (3.13) is positive, while that on the right is negative, so that $\lambda < \eta_1(r_2)$. Hence, the odd solutions $x(\tau, \lambda)$ with amplitudes $A \le \pi/2$ are stable.

According to Theorem 2, in the general case for small amplitudes the solutions $x_1(\tau, \lambda)$ are stable while the solutions $x_2(\tau, \lambda)$ are unstable, and therefore $x(\tau, \lambda) = x_1(\tau, \lambda)$. Hence, the solutions $x_1(t, \omega)$ are odd and the solutions $x_2(t, \omega)$ are even.

We will show that the solution $x_2(t, \omega)$ is unstable. Since $x_2(\tau, \lambda)$ satisfy Eq. (1.10) and the conditions $x'(0, \lambda) = x(\pi, \lambda) = 0$, we have $\lambda = \eta_1(r_1)$. by virtue of relation (1.17) $\eta_1(r_1) < \eta_1(r_2)$ we will show that $\lambda < \mu_1(r_2)$.

As before we obtain

$$(\lambda^2 - \mu_1^2)\int_0^\pi r_2 \upsilon y d\tau = \int_0^\pi (2ll' y \upsilon' - \lambda^2 g l' \upsilon \sin x_2) d\tau \tag{3.14}$$

where $y = x_2'(\tau, \lambda)$, $r_2 = gl(\tau)\cos x_2)(\tau, \lambda)$. In this case the integrand on the right-hand side is negative. By Theorem 3 $y(\tau) = x_2'(\tau, \lambda) < 0$ in $(0, \pi)$, and hence when $|x_2(\tau, \lambda)| \le \pi/2$ the integrand on the left-hand side is also negative. Consequently, $\lambda > \mu_1(r_2)$, and hence, taking into account the inequality $\lambda < \eta_1(r_2)$ we have $\lambda \in (\lambda_1(r_2), \lambda_2(r_2))$.Hence, the solutions $x_2(t, \omega)$ with amplitudes $A_2 \le \pi/2$ are unstable. We will show that the instability is also maintained when $A_2 > \pi/2$. In this case $r_2(\tau) < 0$ in $[0, \tau_*)$ and $r_2(\tau) > 0$ in $(\tau_*, \pi]$, where $x_2(\tau_*, \lambda) = \pi/2$. By virtue of Eq. (3.9)

$$v'(\tau) = -\frac{1}{l^2(\tau)}\int_0^\tau \mu_1^2 r_2(s) v(s) ds > 0 \text{ на } [0, \tau_*) \tag{3.15}$$

This inequality cannot be violated since, if $v'(\tau) = 0$ when $\tau \in (\tau_*, \pi)$, from Eq. (3.9), taking into account the inequality $v(\tau) > 0$, we obtain $v'(\pi) < 0$, which contradicts the boundary condition (3.9). Consequently, when λ increases, the right-hand side of Eq. (3.14) retains its sign and of course, the inequality is maintained, and together with this the instability of the solution $x_2(t, \omega)$. The theorem is proved. □

Note that physically the difference between the solutions considered is as follows. In view of relations (3.6) and (2.14) the length of the pendulum $l(\omega t)$ increases monotonically when $t \in (0, \pi/\omega)$ and decreases when $t \in (\pi/\omega, 2\pi/\omega)$. Since

$$x_1(0, \omega) = 0, \quad x_1(\pi/\omega, \omega) = A_1(\omega)$$

those oscillations of the pendulum for which its length is a minimum at the instant of crossing the equilibrium position and a maximum at the limiting positions correspond to the solution $x_1(t, \omega)$. Conversely, since

$$x_2(0, \omega) = -x(-2\pi/\omega, \omega) = A_2(\omega)$$

for the solution $x_2(t, \omega)$ the length of the pendulum is a minimum at the extreme positions and a maximum in the equilibrium position.

The condition of stability $A_1(\omega) \le \pi/2$ of the solutions $x_1(t, \omega)$ is sufficient; it does not exclude the stability of the solutions with amplitudes $A_1(\omega) > \pi/2$. Nevertheless, it is easy to show that in any case the solutions $x_1(t, \omega)$ with amplitudes close to π, are unstable. In fact, the solution $x_1(\tau, \lambda)$ increases in $(0, \pi)$ and decreases in $(\pi, 2\pi)$; suppose $\tau_1(\lambda) \in (0, \pi)$ and $\tau_2(\lambda) \in (\pi, 2\pi)$ are roots of the equation $x_1(\tau, \lambda) = \pi/2$. As shown when proving Theorem 5, the solution $x_1(\tau, \lambda)$ increases with respect to λ, and hence $\tau_1(\lambda) \to 0$, $\tau_2(\lambda) \to 2\pi$ when $\lambda \to \infty$. It is clear that in the corresponding Eq. (1.11), $r_2(\tau) <$ when $\tau \in (\tau_1, \tau_2)$. As is well known,[8] if $r_2(\tau) < 0$ for all τ, the characteristic constant $A(\lambda) > 1$, i.e. Eq. (1.11) is unstable. It is obvious that the same conclusion also holds for sufficiently small $\tau_1(\lambda)$ and $2\pi - \tau_2(\lambda)$.

For example, when $l(\omega t) = (3 - 0.5\cos\omega t)$ m numerical calculations showed that the solution $x_1(t,\omega)$ is stable when $A_1 < A_1^* = 2.29$ and unstable when $A_1 \in (A_1^*, \pi)$. The solution $x_2(t, \omega)$ is unstable for all ω; this conclusion agrees completely with the assertion of Theorem 6. The amplitude-frequency characteristics of these solutions $A_1(\omega)$ and $A_2(\omega)$ are shown in Fig. 3. In Fig. 4 we show graphs of the functions $l(\tau)$ and also of the functions $x_1(\tau)$ and $x_2(\tau)$ for $\omega = 3.86\ \mathrm{s}^{-1}$ and $\omega = 3.02\ \mathrm{s}^{-1}$ respectively. As can be seen, for the solution $x_1(\tau)$ the length of the pendulum is a minimum when passing through the equilibrium position and a maximum for the solution $x_2(\tau)$.

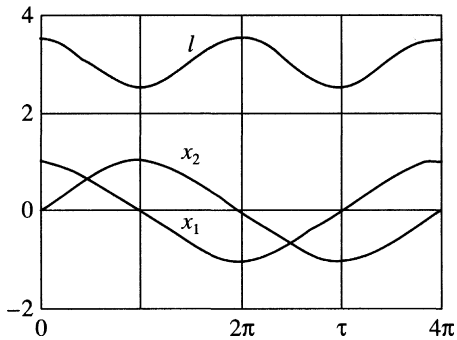

Fig. 4.

4. A mathematical model of a swing

A swing can be represented with a reasonable degree of accuracy by a physical pendulum. When pumping the swing, the rider is squatting when descending and standing when ascending, and as a result the distance L from the centre of gravity of the system to the suspension point varies as a function of the coordinate and the velocity of the swing, i.e. $L = L(x, \dot{x})$. Hence, the oscillations of a swing can be described by the equation

$$[L^2(x,\dot{x})\dot{x}]^{\bullet} + \mu\varphi(x,\dot{x}) + gL(x,\dot{x})\sin x = 0 \tag{4.1}$$

which represents the self-excited oscillatory model of the swing.

Suppose the function $x(t) = x(t + T)$ describes the periodic oscillations of the swing. It is obvious that the function $L(x(t), \dot{x}(t))$ is periodic with period $T/2$, and hence the equation of parametric oscillations of the pendulum (1.1) when $l(t) = L(x(t), \dot{x}(t))$ has the same solution $x(t)$. As a consequence of this analogy, the swing is often sited as an example of the parametric excitation of oscillations and Eq. (1.1) is used to describe a swing. Some authors also assume the parametric and self-excited oscillatory models to be equivalent.

Thus, it has been asserted[3] that a swing "can be regarded as a mathematical pendulum, the length of which varies periodically", and references are made to the corresponding equation of the form (1.1). At the same time, it has been noted further that, since the length of the swing can be expressed by a unique function of the phase coordinates, it "can serve as an example of an oscillator with pure self-excitation" and Eq. (4.1) has been used to investigate the pumping of a swing. The same point of view is expressed in Ref. 4, where the swing (being "the most visual parametric system") is investigated using Eq. (1.1) (Ref. 4, pp. 191–204), but it is then asserted that the oscillations of the swing can also be assumed to be self-excited oscillations (Ref. 4, p. 205).

It is interesting to analyse the relation between the parametric and self-excited oscillatory models of a swing. This problem was considered for the first time in Ref. 10; below it is investigated in more detail, and the theoretical analysis is supplemented by numerical investigations of these models. Note that the discussion below does not relate to non-stationary problems of the pumping and damping of the swing; in the latter the length control law is constructed in the form of a function of the phase coordinates $L = L(x, \dot{x})$ (a rigorous solution of this optimal control problem was obtained in Ref. 4).

It is obvious that different models of the swing can be regarded as equivalent provided they lead to the same qualitative conclusions regarding the properties of their common periodic solution $x(t)$ and, primarily, with regard to its stability. Nevertheless, it turns out that, within the framework of the parametric model, this solution is unstable, which contradicts the obvious stability of the oscillations of actual swings.

In fact, since, when the rider is squatting, when the centre of gravity of the system drops, it is shifted downwards, the length of the swing $L(t)$ reaches its maximum value when it passes through the equilibrium position, whereas at the instant of greatest deflections from the vertical the length is a minimum. As was shown above, this mode of operation corresponds to the solution $x_2(t, \omega)$ of the problem of the parametric oscillations of a pendulum. But, by Theorem 6, this solution is unstable for all ω, which also proves the above assertion.

The reason for this contradiction is that the stability of the solution $x(t)$ of Eq. (1.1) is determined by the equation in variations (1.5), whereas the equation in variations for Eq. (4.1) also contains the derivatives $L(x, \dot{x})$ and $L^2(x, \dot{x})$ with respect to x and with respect to $\dot{x}$, and therefore from a mathematical point of view the use of the parametric model of a swing is equivalent to neglecting these terms. As can be seen, the latter change the character of the stability, and as a result the unstable oscillations of a pendulum correspond to stable oscillations of a swing.

Hence, the mathematical model of a swing in the form of a pendulum with periodically varying length can only be used in the case when the length is a periodic function of time, independent of the phase coordinates of the system (such a model can be realized by placing a robot on a swing with a specified program of periodic squatting). By Theorem 6 those oscillations of a pendulum are stable for which its length at the instant it passes through the equilibrium position is a minimum (the solution $x_1(t)$ corresponds to this); consequently, under steady operating conditions the robot will be standing on descent and squatting on the ascent. As shown above, these oscillations become unstable, beginning from a certain value $A_1^* \in (\pi/2, \pi)$ (whereas one can observe stable oscillations of the swing for any amplitudes $A < \pi$).

We will now analyse the self-excited oscillatory model of a swing. Putting

$$\varphi = \dot{x}, \quad x = x_1, \quad L^2\dot{x} = x_2$$

in Eq. (4.1), we can write it in the form of a system of first-order equations

$$\dot{x}_1 = \frac{x_2}{L^2}; \quad \dot{x}_2 = -gL\sin x_1 - \mu\frac{x_2}{L^2}; \quad L = L(x_1, x_2) \tag{4.2}$$

A general qualitative investigation of system (4.2) is outside the scope of this paper, and hence we will confine ourselves to a numerical analysis. We put

$$L(x_1, x_2) = L_0 + \varepsilon\cos\left[\pi - \frac{\beta\pi|x_2|}{|x_1 + \alpha\operatorname{sgn} x_2| + \beta|x_2|}\right] \tag{4.3}$$

where α and β are small positive constants and ε is the amplitude of the change in the length of the swing with respect to the mean value L_0. Note that the choice of the function $L(x_1, x_2)$ is due to the fact that it agrees qualitatively with the actual law of variation of the length of the swing under steady oscillations, namely: the length is a minimum for the greatest deflections ($L = L_0 - \varepsilon$ when $x_2 = 0$) and a maximum close to the equilibrium position ($L = L_0 + \varepsilon$ for $x_1 = -\alpha$ ($x_2 > 0$) or $x_1 = \alpha$ ($x_2 < 0$)) (the need for a small displacement of the maximum with respect to the equilibrium position $x_1 = 0$ is due to the fact that there is dissipation in the system).

Numerical calculations were carried out for $L_0 = 3$ m, $\varepsilon = 0.5$ m, $\alpha = 0.05$ and $\beta = 0.1$. It was established that the system has one limit cycle with amplitude $A_+ = 0.969$; in the upper part of Fig. 5 we show the corresponding periodic solution $x_1(t)$ and the length $L(t) = L(x_1(t), x_2(t))$. As can be seen, the period $L(t)$ is equal to half the oscillation period of the system, and the minima of $L(t)$ correspond to the minimum and maximum of $x_1(t)$, while the maxima are close to the instants of passage through the equilibrium position. Any solution of system (1.1), (1.5) with initial conditions $x_2(0) = 0$, $x_1(0) \in (0, \pi)$ converges to this cycle; the corresponding solutions $x_1(t)$ for large and small $x_1(0)$ are shown in Fig. 6.

Hence, the self-excited oscillatory model leads to a correct conclusion regarding the stability of the oscillations of a swing.

We also considered the oscillations of the system (4.2), (4.3) for $L_0 = 3$ m, $\varepsilon = -0.5$ m, $\alpha = -0.05$ and $\beta = 0.1$; here the length of the pendulum is a maximum at its greatest deflections and a minimum when $x_1 = -\alpha$ ($x_2 > 0$) or $x_1 = \alpha$ ($x_2 < 0$). It turns out that this system also has a unique limit cycle, and its amplitude $A_- = 0.737$ (the corresponding graphs of $L(t)$ and $x_1(t)$ are shown in the lower part of Fig. 5). Hence, in principle, one can pump on a swing by squatting

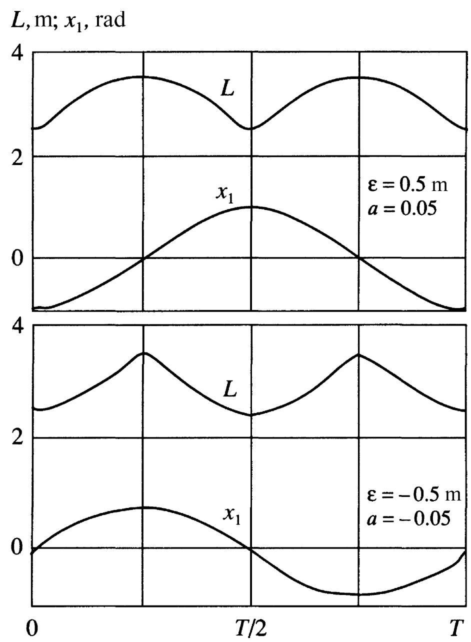

Fig. 5.

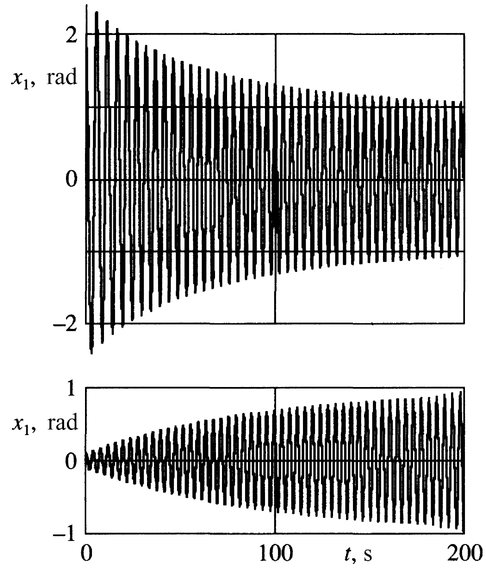

Fig. 6.

on the ascent and standing on the descent. The reason that this mode of operation is not used in practice is obviously the fact that it is natural and easier to squat when the centrifugal force increases (i.e. when moving downwards) and not vice versa.

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